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Pathological Behavior of Renormalization-Group Maps at High Fields and Above the Transition Temperature

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We show that decimation transformations applied to high-q Potts models result in non-Gibbsian measures even for temperatures higher than the transition temperature. We also show that majority transformations applied to the Ising model in a very strong field at low temperatures produce non-Gibbsian measures. This shows that pathological behavior of renormalization-group transformations is even more widespread than previous examples already suggested.

KEY WORDS: Non-Gibbsian measures; real-space renormalization; complete analyticity; majority-rule; decimation.

1. INTRODUCTION

In refs. 29 and 30 it was shown how various renormalization-group (RG) maps acting on Gibbs measures produce non-Gibbsian measures. In physicists' language, this means that a "renormalized Hamiltonian" cannot be defined. The examples presented there were all valid at low temperatures and mostly either in or close to the coexistence region. The underlying

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mechanism—pointed out first by Griffiths, Pearce, and Israel[12, 13, 20]—is the fact that for the constraints imposed by particular choices of block-spin configurations, the resulting system exhibits a first-order phase transition. For this to happen, it was expected that the original system should be itself at or in the vicinity of a phase transition. Block-average transformations, however, provided a counterexample to this belief, in that they lead to non-Gibbsianness for arbitrary values of the magnetic field (at low temperatures).[30]

Since this work was done, there was a sort of "damage-control" movement where various transformations were shown, or argued, to preserve Gibbsianness, or to restore it after sufficiently many iterations. These include sufficiently sparse (or sufficiently often iterated) decimations in nonzero field,[26] possibly combined with other block-spin transformations,[27] decimated projections on a hyperplane,[24] and majority,[21] block-average,[1] and decimation[31] transformations in the (low-temperature) vicinity of the critical point of the two-dimensional Ising model. The case of decimated projections[24] has the peculiarity that the Gibbsianness is restored in a measure-dependent fashion: the renormalized Hamiltonians for the "+" and the "−" Gibbs states are different, and there is no renormalized Hamiltonian for nontrivial mixtures of these states. On the other hand, some studies of the 2d critical Ising model[21, 1, 31] asserting Gibbsianness of some renormalized measures at or in the vicinity of the critical point are highly suggestive, but not conclusive. Indeed, refs. 21 and 31 consider only (judiciously) selected block-spin configurations, while ref. 1 adds a further argument due to Cassandro and Gallavotti stating that the lack of phase transition for the restricted system with block-average spin equal to zero implies Gibbsianness. This argument, however, is still in need of some increased degree of rigor, as one can construct examples for which the original formal argument leads to false conclusions (A. C. D. van Enter, unpublished).

In this paper we present two new examples of non-Gibbsianness that show the ubiquity of this phenomenon of lack of a renormalized Hamiltonian: (1) We show another example of non-Gibbsianness in the *strong-field* region, this time for majority-rule transformations of the Ising model. (2) For the high-q Potts model we show that the decimated measure can be non-Gibbsian for a range of temperature *above* the transition temperature. The first example together with the example of block-averaging[30] show that non-Gibbsianness can appear deep within the region of strong[26, 1] complete analyticity[16], contradicting the intuition explained in refs. 26 and 1. On the other hand, the second example, besides being the first proven example of a "high-temperature" pathology, shows that the condition of strong complete analyticity may be violated above the transition temperature, answering a question posed by Roland Dobrushin. However, it seems plausible that the weaker version of complete analyticity —where only sufficiently regular volumes are involved—applies. This weaker version, discussed, for example, in refs. 26 and 1, is the one which in general seems suitable in the vicinity of first-order transitions.

We mention that Griffiths and Pearce[12,13] and also Hasenfratz and Hasenfratz[17] presented arguments suggesting the existence of "peculiarities" for majority-rule transformations at some precisely tuned (high) values of the magnetic field. Our discussion shows that the situation is even worse than they expected because in fact the pathologies happen for *arbitrarily large* values of the field.

The present examples, in our opinion, support the point of view that the non-Gibbsianness of renormalized measures is in some sense "typical" and should not be dismissed as exceptional. On the other hand, they make even more apparent the need for a systematic study of the consequences of this non-Gibbsianness for computational schemes (renormalization-group calculations, image-processing algorithms) which assume the existence of a renormalized Hamiltonian in the usual sense (see ref. 28 for a pioneer study in this direction).

2. BASIC SET-UP

We consider finite-spin systems in the lattice $\mathscr{L} = \mathbb{Z}^d$, that is, a *configuration space* of the form $\Omega = (\Omega_0)^{\mathbb{Z}^d}$ with the *single-spin space* Ω_0 consisting of a finite set of (integer) numbers. We consider the usual structures: All subsets of Ω_0 are declared to be open (discrete topology) and measurable (discrete σ-algebra), and the normalized counting measure is chosen as the *a priori* probability measure on the single-spin space. The space Ω is endowed with the corresponding product structures. In particular, the product of normalized counting measures acts as an *a priori* probability measure on Ω—the *interaction-free* measure—which we denote μ^0. We shall use a subscript Λ when referring to analogous objects for a subset $\Lambda \subset \mathbb{Z}^d$: for instance, $\Omega_\Lambda \equiv (\Omega_0)^\Lambda$; if $\sigma \in \Omega$, $\sigma_\Lambda \equiv (\sigma_x)_{x \in \Lambda}$, etc. On the other hand, for $\sigma, \omega \in \Omega$ we shall denote by $\sigma_\Lambda \omega$ the configuration equal to σ on sites in Λ and to ω outside.

We point out that, in contrast with the single-spin case, not all subsets of Ω are open, nor are all functions on Ω continuous. In fact, a function $f: \Omega \to \mathbb{R}$ is continuous at σ if and only if

$$\lim_{\Lambda \nearrow \mathscr{L}} \sup_{\omega:\, \omega_\Lambda = \sigma_\Lambda} |f(\sigma) - f(\omega)| = 0 \tag{2.1}$$

that is, a change of σ in faraway sites has little effect on the value of f. That is why continuous functions are, in the present setting, often also called *quasilocal* functions. Here and in the sequel we use the symbol "$\nearrow$" to indicate convergence in the van Hove sense. Also, we point out that the symbol "$|\cdot|$" will also be used to indicate the cardinality of a set.

Each spin model is usually defined in terms of an interaction, that is, a family $\Phi = (\Phi_A)_{A \subset \mathbb{Z}^d,\, A \text{ finite}}$ of functions $\Phi_A: \Omega \to \mathbb{R}$ (contribution of the spins in A to the interaction energy) which are continuous and depend only on the spins in A. These interactions determine the finite-volume Hamiltonians

$$H_\Lambda(\sigma_\Lambda \mid \omega) \equiv \sum_{\substack{\text{finite } A \subset \mathscr{L} \\ A \cap \Lambda \neq \varnothing}} \Phi_A(\sigma_\Lambda \omega) \tag{2.2}$$

and the Boltzmann–Gibbs weights

$$\pi_\Lambda(g \mid \omega) = (\text{Norm.})^{-1} \int g(\sigma_\Lambda \omega) \exp[-H_\Lambda(\sigma_\Lambda \mid \omega)]\, \mu_\Lambda^0(d\sigma_\Lambda) \tag{2.3}$$

In order not to run into problems with the definition of H_Λ and the Boltzmann weights, the usual assumption is that the interactions are *absolutely summable*, i.e., $\sup_x \sum_{A \ni x} \|\Phi_A\|_\infty < \infty$.

The set of Boltzmann weights $\pi(\cdot \mid \cdot)$ form a regular system of conditional probabilities in the sense that they satisfy the "consistency property"

$$\pi_{\tilde{\Lambda}}(\cdot \mid \omega) = \int \pi_\Lambda(\cdot \mid \tilde{\omega})\, \pi_{\tilde{\Lambda}}(d\tilde{\omega} \mid \omega) \tag{2.4}$$

for *all* configurations $\omega \in \Omega$ and all volumes $\Lambda \subset \tilde{\Lambda}$. For this reason, they constitute a system of regular conditional probabilities (for events on finite volumes conditioned on the configurations outside). Moreover, these are conditional probabilities defined for *all* configurations ω, rather than almost all, as is usually the case in probability theory. To emphasize this fact, the term *specification* has been coined.

Specifications defined as in (2.3) are called *Gibbsian specifications*, and they model finite-volume equilibrium for the system in question. The corresponding infinite-volume equilibrium is described by the corresponding *Gibbs measures*, which are those measures μ on Ω whose conditional probabilities are given by the specification

$$\mu(\cdot) = \int \pi_\Lambda(\cdot \mid \omega)\, \mu(d\omega) \tag{2.5}$$

In this case one also says that the measure μ is *consistent* with the specification π. More generally, a probability measure is *Gibbsian* if it is consistent with some Gibbsian specification.

There is an important necessary condition of Gibbsianness: Gibbsian specifications are necessarily continuous—that is, quasilocal—with respect to the boundary conditions. That is [cf. (2.1)], for each finite $\Lambda \subset \mathbb{Z}^d$ and any $\sigma \in \Omega$,

$$\lim_{\Lambda \nearrow \mathscr{L}} \sup_{\omega:\, \omega_\Lambda = \sigma_\Lambda} |\pi_\Lambda(\cdot \mid \sigma) - \pi_\Lambda(\cdot \mid \omega)| = 0 \tag{2.6}$$

with the limit understood in the weak sense (i.e., it holds, possibly at different rates, when "$\cdot$" is replaced by any continuous function depending only on finitely many spins). A measure whose conditional probabilities violate this quasilocality requirement cannot be Gibbsian (see ref. 30 for a more detailed discussion of this issue).

In particular it is of interest to analyze the Gibbsianness of renormalized measures. In its general form, a *renormalization transformation* is a map between probability measures defined by a probability kernel (see ref. 30 for the relevant definitions). In this paper we consider only *deterministic real-space renormalization transformations*. These are defined in the following fashion. One considers a basic "block" B_0—in this paper a cube of linear size N—and paves $\mathbb{Z}^d$ with its translates $\{B_x: x \in N\mathbb{Z}^d\}$ (from now on, whenever we speak about "blocks" we shall mean one of the blocks of a fixed paving). For each block one takes a transformation that associates to each configuration in the block B_x a spin value representing an "effective" block spin. It is mathematically convenient to think of this transformation as going from $\mathbb{Z}^d$ to $\mathbb{Z}^d$, rather than to a "thinned" $\mathbb{Z}^d$, hence we consider maps $T_x: \Omega_{B_x} \to \Omega_0$, defined for each $x \in \mathbb{Z}^d$, and the maps $T: \Omega \to \Omega$ with $[T(\omega)]_x = T_{Nx}(\omega_{B_{Nx}})$ constructed from it. Each such map T defines a renormalization transformation on measures that maps every measure μ on Ω into a new measure $T\mu$, also on Ω, introduced in a natural manner by its action on any measurable function g, namely,

$$\int g(\omega')\, T\mu(d\omega') = \int g(T(\omega))\, \mu(d\omega) \tag{2.7}$$

(As customary, we shall try to use primed variables for the renormalized objects). The two transformations of interest here are odd-block majority-rule transformations for the Ising model ($\sigma_x = +1, -1$),

$$T_x \sigma_{B_x} = \operatorname{sgn}\Big(\sum_{x \in B_x} \sigma_x\Big) \tag{2.8}$$

and decimation for the Potts model,

$$T_x(\sigma_{B_x}) = \sigma_x \tag{2.9}$$

3. NON-GIBBSIANNESS FOR MAJORITY-RULE MAPS OF ISING MODELS AT HIGH MAGNETIC FIELD

We consider the Ising model in $\mathbb{Z}^d$, that is, spins $\sigma_x \in \{-1, 1\}$ with interaction

$$\Phi_A(\sigma) = \begin{cases} -h\sigma_x & \text{if} \quad A = \{x\} \\ -J\sigma_x\sigma_y & \text{if} \quad A = \{x, y\} \quad \text{with } x, y \text{ nearest neighbors} \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

with $J > 0$. The result is the following:

Theorem 3.1 Consider the majority-rule transformation T_L acting on blocks of linear size $2L+1$, $L \geqslant 2$. Let $\mu_{\beta,h}$ denote the unique Gibbs measure for the Ising model at inverse temperature β and magnetic field $h > 0$. Then there exists a β_L such that for $\beta > \beta_L$ and $|h| > J/L$ the measure $T_L\mu_{\beta,h}$ is not consistent with any quasilocal specification; in particular, it is not a Gibbs measure for any uniformly convergent interaction.

For the proof we essentially follow the scheme of ref. 30, Section 4.2: We determine a suitable special configuration ω'_{special} yielding a constrained system with several phases. Let us, for concreteness, consider $h > 0$. In this case we choose ω'_{special} equal to the all-"$-$" configuration, so as to have a constraint acting against the magnetic field. We have to prove two things:

Claim 3.2. The resulting constrained system of internal spins has more than one phase.

Claim 3.3. The different phases of the constrained system can be selected by imposing suitable *block-spin* boundary conditions over a ring-like region of finite width (i.e., by replacing, for this boundary region, the above constraint stemming from ω'_{special} by a different, suitably chosen constraint).

Together these claims imply that by changing block spins arbitrarily far away one changes the phase of the internal spins, which in turns changes the value of block-spin averages close to the origin. For instance, it modifies the (average) value of the block spin at the origin and that of one of its nearest neighbors (when these spins are "unfixed"; this part of the argument is almost identical to the corresponding argument for block-averaging transformations; see Step 3 in ref. 30, pp. 1008–1009). This modification takes place despite the fact that the intermediate block spins are fixed in the configuration ω'_{special}. This means that the *direct* influence of faraway block spins does not decrease with the distance, hence the renormalized measure cannot be Gibbsian.

We emphasize that only block spins on an annulus of *finite width* are invoked in Claim 3.3; the block-spin configurations can be arbitrarily chosen outside it. This implies that there is an "essential" jump in averages of renormalized observables, in which the extremal values of it can be reached via sequences chosen from "large" (nonzero-measure) sets of boundary configurations, obtained by modifying ω'_{special} arbitrarily far away. Mathematically, we are proving that some conditional probabilities of $T_L\mu_{\beta,h}$ are *essentially* discontinuous at ω'_{special}: They exhibit a jump that cannot be removed by redefining them on a set of $\mu_{\beta,h}$-measure zero around ω'_{special}. Hence, no other realization of such conditional probabilities will be free of this discontinuity. Of course, one may attempt to do without ω'_{special}; after all, conditional probabilities need to be defined only $T_L\mu_{\beta,h}$-almost everywhere. This is a more involved issue, about which we shall briefly comment in Section 5. The finiteness of the annulus in Claim 3.3 is needed for a second reason: *A priori* we only know that the conditional probabilities of $T_L\mu_{\beta,h}$ are *some* Gibbs states of the constrained system of internal spins [see the discussion of Step 0 (esp. pp. 987–990) in ref. 30], but we do not know which ones. Therefore, the statements have to be proved for *all* possible such Gibbs states, which is equivalent (ref. 10, Theorem 7.12) to proving them for *arbitrary boundary conditions* (see ref. 30, p. 991, for a more complete discussion of these issues).

We discuss the proof of the claims above only in the particular case of $d=2$ and $L=2$ (5×5-blocks). The other cases are analogous, but they require a more complicated accounting of ground states that would obscure the argument.

3.1. Proof of Claim 3.2

We start by analyzing the ground-state configurations of the constrained system. These configurations must satisfy the constraint of keeping each block with a majority of "$-$" while maximizing the number of spins parallel to the field and minimizing the number of "$+$"-"$-$" pairs (broken bonds). This clearly yields, inside 5×5 blocks and for $h>J$, the eight ground-state configurations shown in Fig. 1. Any overall ground-state configuration combines such blocks without any interruption. It is easy to convince oneself that there is an infinite number of such ground-state configurations and that this set splits into four classes consisting of configurations with either horizontal or vertical alternating strips as depicted in Fig. 2. Within each strip a primed block always neighbors an unprimed one and one has the freedom to start, in each strip independently of the other strips, with the primed or unprimed one. This yields two possible arrangements [mapped one into another by a shift by one (block) lattice spacing]

1	1′	2	2′
+ + + + +	+ + + + +	− − − − −	− − − − −
+ + + + +	+ + + + +	− − − − −	− − − − −
− − − + +	+ + − − −	− − − + +	+ + − − −
− − − − −	− − − − −	+ + + + +	+ + + + +
− − − − −	− − − − −	+ + + + +	+ + + + +

3	3′	4	4′
+ + − − −	+ + + − −	− − − + +	− − + + +
+ + − − −	+ + + − −	− − − + +	− − + + +
+ + − − −	+ + − − −	− − − + +	− − − + +
+ + + − −	+ + − − −	− − + + +	− − − + +
+ + + − −	+ + − − −	− − + + +	− − − + +

Fig. 1. Configurations minimizing the energy within a 5×5-block for the Ising model with negative block magnetization in the regime $h > J$.

for each strip and leads to the degeneracy of the order $2^{\text{number of strips}}$ of each of these classes of ground-state configurations.

We assert that each class of ground-state configurations gives rise to a different low-temperature Gibbs measure. In such measures only the identity of the class is kept—the periodic long-range order between primed and unprimed blocks present in particular ground-state configurations is not conserved at nonvanishing temperatures, as it is, effectively, a one-dimensional order. The proof of this assertion, from which Claim 3.2 follows, can be done in (at least) two different ways.

I

1′	1	1′	1	1′	1
2	2′	2	2′	2	2′
1	1′	1	1′	1	1′
2′	2	2′	2	2′	2
1′	1	1′	1	1′	1
2′	2	2′	2	2′	2

II

2′	2	2′	2	2′	2
1′	1	1′	1	1′	1
2	2′	2	2′	2	2′
1	1′	1	1′	1	1′
2′	2	2′	2	2′	2
1	1′	1	1′	1	1′

III

3′	4	3′	4	3	4′
3	4′	3	4′	3′	4
3′	4	3′	4	3	4′
3	4′	3	4′	3′	4
3′	4	3′	4	3	4′
3	4′	3	4′	3′	4

IV

4′	3	4	3′	4′	3
4	3′	4′	3	4	3′
4′	3	4	3′	4′	3
4	3′	4′	3	4	3′
4′	3	4	3′	4′	3
4	3′	4′	3	4	3′

Fig. 2. Classes of ground states for the Ising model with negative block magnetization (5×5-block, $h > J$). Within each strip the primed blocks can be at either odd or even positions, independent of the configuration in other strips.

The first one is to use chessboard estimates in the form presented in Theorem 18.25 of ref. 10. Indeed, by considering each block as a single-spin space with as many values as block configurations satisfying the constraint of having a majority "−" we can map our constrained system into an unconstrained one with $|\Omega_0| = 2^{24}$ and with a certain one- and two-body nearest-neighbor interaction. This systems is clearly reflection-positive and the four classes of Fig. 2 are just the classes $G_1, \ldots, G_4$ of the above-mentioned theorem.

One can also prove the existence of four low-temperature Gibbs states with the help of the generalization of Pirogov–Sinai theory due to Bricmont, Kuroda and Lebowitz (BKL) in ref. 3. Let us briefly review BKL theory, as we also will apply it later for the example of the Potts model. The central objects of the theory are the *restricted ensembles* which are families or classes of configurations that play a role analogous to that of the ground states in the standard Pirogov–Sinai theory. In the BKL version, the restricted ensembles have a product structure: they are characterized by their configurations on an elementary cube C_0. More precisely, Ω_{C_0} can be partitioned,

$$\Omega_{C_0} = \left[\bigcup_{a=1}^{r} \Omega_0^a \right] \cup \bar{\Omega}_0 \tag{3.2}$$

with each Ω_0^a associated to a restricted ensemble and $\bar{\Omega}_0$ containing what is left. By paving the lattice with translates C_x of C_0 with $x \in L\mathbb{Z}^d$, where L is the linear size of C_0, one defines the translated cube configurations Ω_x^a. The ath restricted ensemble is formed by configurations whose restriction to each C_x is of the type Ω_x^a:

$$\Omega^a = \{\sigma \in \Omega : \sigma_{C_x} \in \Omega_x^a \text{ for all } x \in L\mathbb{Z}^d\} \tag{3.3}$$

For each restricted ensemble one considers the corresponding restricted partition functions in finite volumes Λ,

$$Z_R(\Lambda, \omega^a) = \sum_{\sigma_\Lambda \in \Omega_\Lambda^a} \exp[-H_\Lambda(\sigma_\Lambda \mid \omega^a)] \tag{3.4}$$

with boundary conditions $\omega^a \in \Omega^a$.

To apply BKL theory, several hypotheses must be satisfied [hypotheses (A1)–(A5) in ref. 3]. First, there is the *diluteness hypothesis*, which basically means that the restricted partition functions must admit a polymer expansion from which a convergent cluster (high-temperature, Mayer) expansion follows. The diluteness hypothesis implies, in particular, that the restricted free energies

$$f^a \equiv -\lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z_R(\Lambda, \omega^a) \tag{3.5}$$

exist and are independent of the choice of $\omega^a \in \Omega^a$. Second, one assumes a restricted-ensemble *Peierls condition*, i.e., that the free-energy cost of placing a droplet of configurations of one of the restricted ensembles inside a sea corresponding to another restricted ensemble be proportional to the surface of the droplet. An important role is played by the value τ of the constant of proportionality. Third, the system must exhibit *free-energy* degeneracy among the restricted ensembles:

$$f^a = f^b, \qquad 1 \leqslant a, b \leqslant r \tag{3.6}$$

If restricted ensembles are formed by exactly one configuration, then the restricted free energies are just energy densities; in that case (3.6) is the usual degeneracy condition of ground states. BKL also assumes the existence of $r-1$ sufficiently smooth perturbations of the interaction, modulated by parameters $\underline{\mu} = (\mu_1, ..., \mu_{r-1})$, which are degeneracy lifting in the sense that the perturbed restricted free energies $f^a_{\underline{\mu}}$ produce a phase diagram that obeys the Gibbs phase rule. More explicitly, the manifolds in $\underline{\mu}$-space defined by inequalities of the form

$$f^{a_1}_{\underline{\mu}} = \cdots = f^{a_k}_{\underline{\mu}} < f^{a_{k+1}}_{\underline{\mu}}, ..., f^{a_r}_{\underline{\mu}}$$

("manifolds of k-phase coexistence") can be homeomorphically mapped, for $\underline{\mu}$ small enough, onto an $(r-k)$-dimensional hypersurface of the boundary of the positive r-octant in $\mathbb{R}^r$. In particular $\underline{\mu} = \underline{0}$ is the only value for which all the restricted free energies coincide.

Under these hypotheses, the conclusion of BKL theory is that *for τ large enough* the actual phase diagram of the system is only a small perturbation of the one drawn with the restricted free energies. In particular there is a value $\underline{\mu}_0$ of the parameters for which all r phases associated to the respective restricted ensembles coexist. Moreover, this coexistence happens for

$$\|\underline{\mu}_0\|_\infty < \text{const} \cdot e^{-\tau} \tag{3.7}$$

that is, the distance between the true maximal-coexistence point and the one determined via the restricted ensembles by (3.6) tends exponentially to zero with the Peierls constant. The typical configurations of the different Gibbs states are formed by an infinite sea of spins configured as in the corresponding restricted ensemble, with small bubbles here and there configured as in the other ensembles.

It is clear how to apply BKL theory for the case of interest here: The restricted ensembles are the four classes $\Omega^{\mathrm{I}}, ..., \Omega^{\mathrm{IV}}$ obtained from the corresponding configurations of Fig. 2 by allowing a free assignment of the primes. Notice that we extend the original classes of ground configurations by ignoring the (fake) one-dimensional primed-unprimed order. In spite of the fact that restricted excitations are included, the classes keep identity and, in particular, the Peierls condition may be verified. For each restricted ensemble, the restricted partition function is (can be put in correspondence with) a product of partition functions for one-dimensional antiferromagnetic Ising models with nearest neighbor coupling $-J$ (the "primes" of different lines do not interact, and two consecutive primes or two consecutive non-primes along a line cost an energy $2J$). The partition functions for one-dimensional finite-range systems have all the diluteness properties in the world, and the four classes have the same restricted free energy density. Explicitly, one can easily verify the diluteness hypothesis in the alternative formulation from Section 4 of ref. 3, that is, by exhibiting an exponential decay of truncated correlations.

To verify the Peierls condition, one has to evaluate the ratio

$$Q(\Gamma \mid \Lambda, \omega^a) = \frac{Z(\Gamma \mid \Lambda, \omega^a)}{Z_R(\Lambda, \omega^a)} \tag{3.8}$$

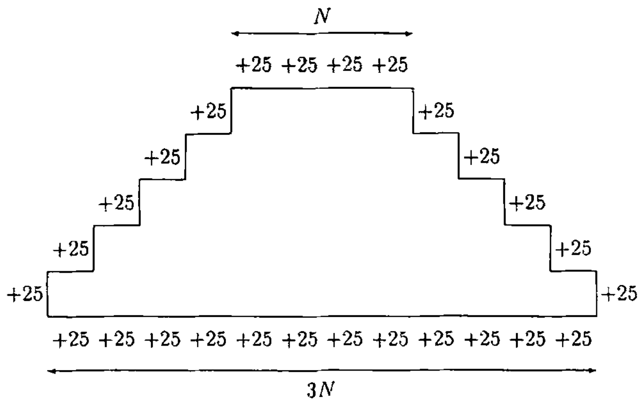
with $Z(\Gamma \mid \Lambda, \omega^a)$ denoting the partition functions obtained by summing over all configurations in Λ having only one contour Γ (the union of blocks that differ from the minimizing ones shown in Fig. 1 equals Γ). Using the above-mentioned effective equivalence of the restricted ensemble with uncoupled one-dimensional Ising models, we evaluate (up to boundary terms) the restricted partition function $Z_R(\Lambda, \omega^a)$ by $(1 + e^{-2\beta J})^{|\Lambda|}$. Noticing that every block in Γ is disfavored by at least the factor $e^{-2\beta J}$, we get the Peierls condition with the Peierls constant being at least $\tau \geqslant 2\beta J$. As symmetry-breaking perturbations we can take fields selecting one or the other of the classes. BKL theory implies, therefore, that for low enough temperature there is a set of values for these fields (not exceeding $e^{-2\beta J}$) at which four Gibbs states coexist which are supported on configurations that, except for small fluctuations, look like those of the corresponding restricted ensemble. Symmetry considerations imply that these coexistence points actually occur when all the perturbing fields vanish.

This argument proves Claim 3.2, and constitutes the rigorous version of the stated breaking of the long-range order between primed and unprimed blocks.

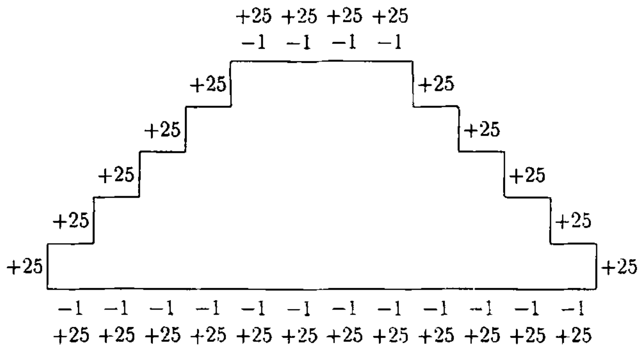
3.2. Proof of Claim 3.3

We start by noticing that if volumes Λ as in Fig. 3 had internal-spin boundary configurations as in part (a) of the figure [resp. part (b)], then the limit $\Lambda \nearrow \mathbb{Z}^2$ would select the Gibbs measure corresponding to the class labeled I [resp. II] in Fig. 2. This can be seen through a small adaptation of the usual Peierls argument: the left and right diagonals are "neutral" in that they do not favor any of the ground states, while the top and bottom favor class I over II in case (a), and conversely in case (b). Similarly chosen rotated volumes select classes III and IV.

However, we are allowed to impose only *block-spin* configurations, which determine the internal spins only in a probabilistic sense. We have



(a)



(b)

Fig. 3. Internal-spin configurations that would select the Gibbs measure corresponding to ground states (a) of class I (Fig. 2), (b) of class II. Numbers indicate total block magnetization. Note: These pictures illustrate the boundary conditions; the number of rows (lateral "steps") should be even and not odd as depicted here. (We thank Marek Biskup for pointing out this inaccuracy to us.)

to prove that there exist some block-spin configurations which, when imposed on some annulus of *finite* radius around Λ, produce *with high probability* the internal-spin configurations of Fig. 3. As the reader may suspect, such a configuration will be the all-"+" block-spin configuration for case (a) (Fig. 4a). For case (b) we shall consider the configuration of Fig. (4b). Let us discuss the former case; the latter is just a slightly modified version of it. The argument is basically a combination of Steps 2.1–2.4 of ref. 30 (cf. p. 1005 there) and well-known probabilistic Peierls arguments (see, for instance, ref. 4, Section 2).

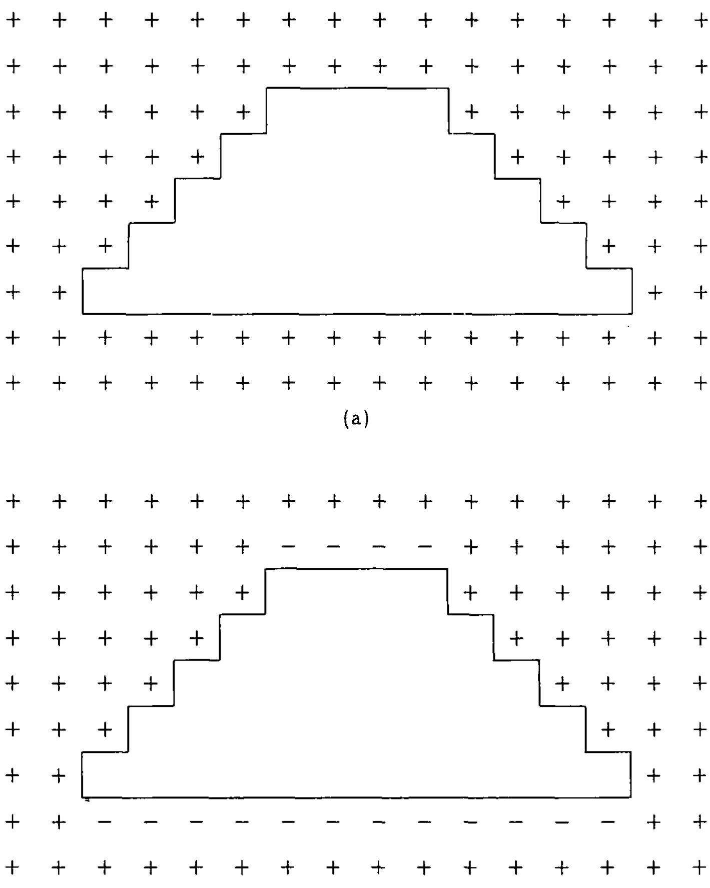


Fig. 4. Block-spin configurations that yield, with high probability, the internal-spin configurations of Fig. 3.

The precise statements require further notation. For a block B, denote

$$N_+(B) = \text{number of "+" spins in } B \tag{3.9}$$

For any family γ of 5×5-blocks we use $|\gamma|$ to denote the number of blocks in γ (for a given configuration) and take

$$\mathscr{B}(\gamma) = \{\text{blocks } B \in \gamma\colon N_+(B) < 25\} \tag{3.10}$$

as the set of blocks of γ with "bad" internal-spin configurations. For volumes V formed by a union of nonoverlapping blocks we consider the probability measures $\hat{\pi}_V^+(\cdot \mid \sigma)$ obtained from the Ising specification with the additional restriction that there must be a majority of "+" spins within each block in V. In an analogous way we define the finite-volume measures $\hat{\pi}_V^{+|-}(\cdot, \sigma)$ with the blocks inside Λ having a majority of "−" spins and those outside a majority of "+" spins.

We decompose now the argument yielding the proof of Claim 3.3 into a sequence of rather natural observations:

Observation 3.4. There is a unique measure $\hat{\mu}^+$ consistent with the specification $\{\hat{\pi}_V^+\}$. Likewise, for a fixed, finite union of blocks Λ, there is a unique measure $\hat{\mu}^{+|-\Lambda}$ consistent with the specification $\{\hat{\pi}_V^{+|-\Lambda}\}$.

Indeed, the uniqueness of $\hat{\mu}^+$ (at all temperatures) follows from the ferromagnetic nature of the model and the uniqueness of the ground state: The latter implies via Griffiths II inequality,[11] that for each temperature the expectations with "+" boundary conditions are equal to those with "−" boundary conditions. This implies uniqueness by FKG-type arguments,[4]. The uniqueness of $\hat{\mu}^+$ implies that of $\hat{\mu}^{+|-\Lambda}$ because the distributions $\{\hat{\pi}_V^{+|-\Lambda}\}$ are only a finite-volume modification of the kernels $\{\hat{\pi}_V^+\}$ (ref. 10, Section 7.4).

Observation 3.5. There exists a constant c such that, for $h > J/2$,

$$\hat{\pi}_B^{+|-\Lambda}(N_+(B) = 25 \mid -) \geqslant 1 - ce^{-\beta h} \tag{3.11}$$

for any block B outside Λ.

This is just the fact that, for $h > J/2$, a block with less than 25 spins "+" (but with at least 13 pluses) has, under minus boundary conditions, an energy cost of least βh. The constant c is just the number of configurations of such a block, $c = 2^{24} - 1$.

Observation 3.6. For each $\delta > 0$ there exists a constant $\tilde{\beta}$ such that for $\beta > \tilde{\beta}$ and $h > J/2$

$$\hat{\mu}^{+|-\Lambda}(|\mathscr{B}(\gamma)| > \delta\,|\gamma|) \leqslant \varepsilon^{|\gamma|} \tag{3.12}$$

with $\varepsilon < 1$, for all families γ of 5×5-blocks located *outside* Λ.

This is proven via the well-known technique of the Bernstein, or "exponential Chebyshev," inequality.[2, 19] To simplify the notation, let us define a block-random variable

$$X_B = \begin{cases} 1 & \text{if } N_+(B) < 25 \\ 0 & \text{otherwise} \end{cases} \tag{3.13}$$

We then have

$$\begin{aligned} \hat{\mu}^{+|-\Lambda}(|\mathscr{B}(\gamma)| > \delta\,|\gamma|) &\leqslant \hat{\mu}^{+|-\Lambda}\left(I\left[\sum_{B\in\gamma} X_B > \delta\,|\gamma|\right] \exp\left[\sum_{B\in\gamma} X_B - \delta\,|\gamma|\right]\right) \\ &\leqslant \hat{\mu}^{+|-\Lambda}\left(\exp\left[\sum_{B\in\gamma} X_B - \delta\,|\gamma|\right]\right) \end{aligned} \tag{3.14}$$

(In the first inequality, $I[A]$ is the indicator function of the event A). By FKG inequalities and Observation 3.5,

$$\begin{aligned} \hat{\mu}^{+|-\Lambda}\left(\exp\left[\sum_{B\in\gamma} X_B - \delta\,|\gamma|\right]\right) &\prod_{B\in\gamma} [e^{-\delta}\hat{\pi}_B^{+|-\Lambda}(e^{X_B}|-)] \\ &\leqslant [e^{-\delta}(1 + ce^{-\beta h}e)]^{|\gamma|} \\ &\equiv \varepsilon^{|\gamma|} \end{aligned} \tag{3.15}$$

Observation 3.7 There exists a constant β_2 such that for $\beta > \beta_2$ and $h > J/2$ the blocks close to the origin have $\hat{\mu}^{+|-\Lambda}$-probability larger than 1/2 to be in the configuration of the ground states of class I (Fig. 2).

This follows from the preceding observation by a probabilistic Peierls argument. Take $\gamma = \partial\Lambda$, that is, equal to the blocks immediately outside Λ, and $\delta = 1/18$. Then by Observation 3.6 there is a very large probability that the configuration on $\partial\Lambda$ looks like that in Fig. 3a except for a small fraction of "bad" blocks that does not exceed one-third of the blocks in the smallest side of Λ (because we chose $\delta = 1/18$; see dimensions in Fig. 3). In this situation, a standard Peierls argument, as sketched at the beginning of the proof of the claim, yields the above observation. The contribution due to configurations of $\partial\Lambda$ with a larger fraction of "bad" blocks is bounded by $\varepsilon^{|\partial\Lambda|}$, which tends to zero as Λ grows.

Observation 3.8. For any configuration σ one has

$$\lim_{V \nearrow \mathbb{Z}^2} \hat{\pi}_V^{+|-\Lambda}(\cdot \mid \sigma) = \hat{\mu}^{+|-\Lambda}(\cdot) \tag{3.16}$$

(in the weak sense).

Indeed, every accumulation point of sequences (nets) $\hat{\pi}_{V_n}^{+|-\Lambda}(\cdot \mid \sigma^{(n)})$ is a Gibbs state of the specification $\{\hat{\pi}_V^{+|-\Lambda}\}$ (it is easy to see that such accumulation points must satisfy the corresponding DLR equations), but by Observation 3.4 there is only one such a Gibbs state, namely $\hat{\mu}^{+|-\Lambda}$.

The last observation implies that we can replace $\hat{\mu}^{+|-\Lambda}$ by $\hat{\pi}_V^{+|-\Lambda}(\cdot \mid \sigma)$ in Observation 3.7. This proves Claim 3.3.

The proof of Theorem 3.1 can now be completed almost identically to the proof for block-average transformations in ref. 30: Claims 3.2 and 3.3 constitute Steps 1 and 2 respectively, and one can then proceed to Step 3 ("unfixing" of the block spins close to the origin) as in pp. 1008–1009 of ref. 30. The conclusion is that there exists a (van Hove) sequence of volumes $\Lambda \nearrow \mathbb{Z}^d$ (those shown in Fig. 3) and open sets of (block-spin) configurations $\mathcal{N}'_+$ ("+" on an annulus surrounding Λ and arbitrary otherwise) and $\mathcal{N}'_-$ ("thickened version of those of Fig. 4b: "$-$" immediately above and below Λ, then an annulus of "+" and arbitrary farther out), such that there exists a constant $c > 0$, *independent of* Λ, with

$$\begin{aligned} |E_{T_L\mu_{\beta,h}}(\sigma'_0 + \sigma'_{e_1} \mid \{\sigma'_x\}_{x \neq 0, e_1})(-'_\Lambda \eta') \\ - E_{T_L\mu_{\beta,h}}(\sigma'_0 + \sigma'_{e_1} \mid \{\sigma'_x\}_{x \neq 0, e_1})(-'_\Lambda \theta')| > c \end{aligned} \tag{3.17}$$

for every $\eta' \in \mathcal{N}'_+$ and $\theta' \in \mathcal{N}'_-$. We have denoted $e_1 = (0, 1)$ and $\omega'_\Lambda \eta'$ is the configuration equal to ω' inside Λ and to η' otherwise. That is, $T_L\mu_{\beta,h}$ has a conditional probability which is essentially discontinuous at $\omega'_{\text{special}} = $ "$-$". In particular, it cannot be Gibbsian.

4. NON-GIBBSIANNESS OF DECIMATED POTTS MODELS ABOVE THE TRANSITION TEMPERATURE

We consider now the q-state Potts model in $\mathbb{Z}^d$, which is defined by spins $\sigma_x \in \{1, ..., q\}$ and interaction

$$\Phi_A(\sigma) = \begin{cases} -J(\delta(\sigma_x, \sigma_y) - 1) & \text{if } A = \{x, y\} \text{ with } x, y \text{ nearest neighbors} \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

and suppose that $J > 0$. Here $\delta(\sigma_x, \sigma_y)$ equals 1 if $\sigma_x = \sigma_y$ and 0 otherwise. To simplify the notation, we incorporate in the following the coupling J into the inverse temperature β [i.e., we put $J = 1$ in (4.1)]. Below we shall also refer to the corresponding model with a field in the 1-direction. By that we mean the addition of interaction terms $h_x \delta(\sigma_x, 1)$ at each $x \in \mathbb{Z}^d$.

For $q=2$ the Potts model becomes (equivalent to) the Ising model. On the other hand, for large q very different properties emerge; in particular, it is known that for q sufficiently high the Potts model exhibits a first-order phase transition[22, 3] with critical inverse temperature

$$\beta_c = \frac{1}{d} \ln q + O\left(\frac{1}{q}\right) \tag{4.2}$$

Our results apply to models with q sufficiently high, and we find it useful to present them in three steps of increasing technical complication.

4.1. Lack of Complete Analyticity Above T_c

As a warm-up step we shall show the following:

Theorem 4.1. If q is sufficiently high and the spins of the sublattice $(N\mathbb{Z})^d$ are fixed to be equal to 1, the resulting system on the rest of the lattice has a first-order phase transition at a temperature $T_c^{(N)}$ which is strictly larger than the Potts critical temperature T_c.

This theorem can be interpreted as showing that at $T_c^{(N)}$ one can find sequences of volumes [those with "holes" at the sites in $(N\mathbb{Z})^d$ and boundary conditions equal to 1 at the holes and 1 or disordered at other boundaries] yielding, in the limit, different one-sided derivatives of the free energy density. In particular, this means that the analyticity of the (finite-volume) free energies cannot be uniform in the volume and the boundary conditions; that is, there is no (strong) complete analyticity.

We will prove Theorem 4.1 by transcribing the proof by Bricmont, Kuroda, and Lebowitz (ref. 3, Theorem 5) of the existence of a first-order phase transition for the regular Potts model. Before doing so, however, let us briefly show the main ideas of an alternative proof based on the use of chessboard estimates. To minimize technicalities, we will restrict ourselves here to the case of $N=2$. The proof is particulary simple if one uses reflection positivity with respect to (hyper)planes passing through the sites of the lattice (see ref. 5 for the details of the use of this particular version of chessboard estimates for the Potts model). In accordance with the standard use of the method, one has to evaluate the "partition functions" $Z^P(T)$ corresponding to the patterns obtained on a torus T by disseminating, with the help of reflections, particular patterns P on a single elementary (hyper)-cube C containing 2^d lattice sites. All then boils down to the verification of the bounds claiming that the patterns stemming from completely disordered configurations on C as well as from the configuration with all spins fixed to equal 1 are dominating over all remaining patterns. Recalling that the spins on the sublattice $(2\mathbb{Z})^d$ are fixed to equal 1, the first two patterns yield the partition functions

$$Z^{\text{disorder}}(T) \sim (q^{(2^d-1)/2^d} e^{-d\beta J})^{|T|} \qquad \text{and} \qquad Z^1(T) = 1$$

respectively. For any other pattern, one easily finds

$$\frac{Z^P(T)}{\max(Z^{\text{disorder}}(T), Z^1(T))} \leqslant \varepsilon \tag{4.3}$$

with sufficiently small ε. Indeed, considering for simplicity the two-dimensional case, we take as an example the pattern stemming from the situation where the horizontal bond attached to the chosen site on $(2\mathbb{Z})^2 \cap C$ is ordered and all remaining (three) bonds in C are disordered. It yields the pattern with every horizontal line through sites in $(2\mathbb{Z})^2$ ordered (all sites at any such line are set to equal 1) and with all remaining bonds disordered. As a result we get $Z^P(T) \sim (q^{1/2} e^{-3\beta J/2})^{|T|}$ and thus (4.3) is satisfied for all β once q is large enough. (Namely, we have here $\varepsilon = q^{-1/16}$.) To show that the transition temperature is asymptotically behaving like

$$\beta_c \sim \frac{2^d - 1}{2^d} \frac{1}{d} \log q$$

one has just to notice that it is exactly this value of β for which $Z^{\text{disorder}}(T) = Z^1(T)$. Hence, for large q, slightly below β_c the disordered pattern dominates also the ordered one, while slightly above β_c, it is the ordered pattern that is dominating.

Coming back to the proof using the BKL theory (reviewed in Section 3), we again use the fact that Theorem 4.1 refers to a Potts model on $\mathbb{Z}^d \backslash (N\mathbb{Z})^d$ with a magnetic field in the 1-direction of strength $h_x = 1$ if x is adjacent to the sublattice $(N\mathbb{Z})^d$ and zero otherwise. One can then choose the "restricted ensembles" Ω^D and Ω^1 formed, respectively, by the disordered and the "all-1" configurations:

$$\Omega^D = \{\sigma : \sigma_x \neq \sigma_y \text{ for all } x, y \text{ nearest neighbors in } \mathbb{Z}^d \backslash (N\mathbb{Z})^d$$
$$\text{and } \sigma_x \neq 1 \text{ for } x \text{ adjacent to } (N\mathbb{Z})^d\} \tag{4.4}$$

and

$$\Omega^1 = \{1\} \tag{4.5}$$

where $1_x = 1$ for all $x \in \mathbb{Z}^d \backslash (N\mathbb{Z})^d$. For each of these ensembles one constructs restricted partition functions; for instance, for any $\omega \in \Omega^D$, we take

$$\begin{aligned} Z_R^D(\Lambda, \omega) &\equiv \sum_{\sigma_\Lambda :\, \sigma_\Lambda \omega \in \Omega^D} e^{-\beta H_\Lambda(\sigma \mid \omega)} \\ &= |\{\sigma_\Lambda \in \Omega_\Lambda : \sigma_\Lambda \omega \in \Omega^D\}| \, e^{-\beta H_\Lambda^D} \\ &\equiv e^{S_\Lambda(\omega)} e^{-\beta H_\Lambda^D} \end{aligned} \tag{4.6}$$

The notation of the last line emphasizes the fact that the term $H_\Lambda(\sigma \mid \omega) \equiv H_\Lambda^D$ does not depend on the configurations σ and ω once $\sigma_\Lambda \omega$ belongs to Ω^D and as a result we can separate the entropy term $S_\Lambda(\omega)$. Notice also that even though, strictly speaking, the entropy $S_\Lambda(\omega)$ depends on a particular choice of $\omega \in \Omega^D$, this dependence is asymptotically negligible [cf. (4.9) below]. On the other hand,

$$Z_R(\Lambda, 1) \equiv 1 \tag{4.7}$$

The system with restricted ensembles (4.4) and (4.5) and restricted partition functions (4.6) and (4.7) satisfies the requirements (A1)–(A5) of ref. 3 just as the usual Potts model does (pp. 522–524 of ref. 3). In particular, the Peierls condition holds with

$$e^{-\tau} \propto \frac{1}{q} \tag{4.8}$$

and the symmetry-breaking parameter is $\beta - \beta_0$, where β_0 is the approximate coexistence temperature obtained via restricted ensembles. (Hence, $1/q$ plays here the role that the temperature plays in the usual Pirogov–Sinai theory, while the temperature plays the role of a field.) By the BKL extension of Pirogov–Sinai theory, we conclude that there is a temperature where the disordered and "all-1" phases coexist. Moreover, by (3.7) and (4.8), we have that, up to corrections of order $1/q$, the transition temperature is determined by the equality of the restricted free energy densities, that is, by the relation

$$\lim_{\Lambda \nearrow \mathbb{Z}^d \backslash (N\mathbb{Z})^d} \frac{S_\Lambda(\omega)}{|\Lambda|} = \lim_{\Lambda \nearrow \mathbb{Z}^d \backslash (N\mathbb{Z})^d} \beta H_\Lambda^1 \tag{4.9}$$

The limiting value of the left-hand side in (4.9) actually does not depend on a particular choice of $\omega \in \Omega^D$. To construct a disordered configuration, the number of choices per site is at least $q - 2d$ (assuming all the neighboring spins have been chosen) and at most q. Hence,

$$S_\Lambda(\omega) = |\Lambda| \, [\ln q + O(1/q)] \tag{4.10}$$

On the other hand,

$$H_A^1 = |A|\, d\left(1 + \frac{1}{N^d - 1}\right) + O(|\partial A|) \tag{4.11}$$

where the term $d\,|A|/(N^d - 1)$ is due to the interaction between spins in A and spins on the decimated sublattice $\mathbb{Z}^d \backslash (N\mathbb{Z})^d$. From (4.9)–(4.11) we get

$$\beta_c^{(N)} = \frac{N^d - 1}{N^d} \frac{1}{d} \ln q + O\left(\frac{1}{q}\right) \tag{4.12}$$

which, for large q, is smaller by a factor $(N^d - 1)/N^d$ than the Potts inverse critical temperature (4.2).

4.2. Non-Gibbsianness for a Sequence of Temperatures Above T_c

Theorem 4.1 amounts to proving what in ref. 30 (see, e.g., p. 990) was referred to as Step 1 of the proof of non-Gibbsianness (more precisely, nonquasilocality) of the renormalized measure. Such a version of Step 1, however, cannot be extended to a full proof of non-Gibbsianness because ω'_{special} is a "maximal" block-spin configuration, and hence there is no way to select the different (internal-spin) pure phases just via block-spin boundary configurations (that is, Step 2 fails). This type of difficulty is already present in other expected examples of non-Gibbsianness proposed in the literature (see discussion on pp. 1006–1007 of ref. 30).

To circumvent this problem, one must prove the analogue of Theorem 4.1 but for decimated spins fixed in some nonuniform configuration. This is easily accomplished: take a periodic configuration in $\mathbb{Z}^d \backslash (N\mathbb{Z})^d$ with a fraction $f < 1/2$ of spins chosen to equal 2 and the rest to equal 1. The same arguments as in the previous section apply, except that (4.11) is generalized to

$$H_A^1 = |A|\, d\left(1 + \frac{1 - 2f}{N^d - 1}\right) + O(|\partial A|) \tag{4.13}$$

hence the coexistence between the "all-1" and disordered phases takes place at an inverse temperature

$$\beta_c^{(N, f)} = \frac{N^d - 1}{N^d - 2f} \frac{1}{d} \ln q + O\left(\frac{1}{q}\right) \tag{4.14}$$

As a result, we now have two phases that can be selected via decimated-spin boundary conditions: if such spins are chosen to be 1, then the "all-1" phase is singled out; and any choice disfavoring it, for instance, boundary-decimated spins chosen equal to 3, selects the disordered phase (Step 2 of ref. 30). The argument can be completed as for decimation of Ising spins (Step 3 in ref. 30) to prove the discontinuity of the decimated conditional probabilities at the inverse temperatures $\beta_c^{(N,f)} < \beta_c$. We notice that for fixed N (decimation scheme), these inverse temperatures range from $\beta_c^{(N)}$ of the previous section (for $f = 0$) and the Potts model β_c given in (4.2) (for $f = 1/2$). As discussed in the previous section, our proof of non-Gibbsianness does not apply for $f = 0$. It does, however, apply at $f = 1/2$, where at the corresponding critical temperature there are *three* coexisting phases: "all-1", "all-2", and disordered.

On the other hand, the term $O(1/q)$ in (4.14) is *not* uniform in the period of the decimated configuration chosen. In fact, a closer look at the proof of Bricmont, Kuroda, and Lebowitz reveals that the larger the period, the larger the minimal value of q needed. Hence, for each fixed q (and N), there is only a finite set of qualifying fractions f, that is, the argument yields only a *finite* sequence of critical inverse temperatures.

We summarize the results of this section:

Theorem 4.2. For each dimension $d \geqslant 2$ and each decimation of period N there exists a q_0 such that for each $q > q_0$ there exists a finite sequence of temperatures $\{T_c^{(N,f(q))}\}$, $f(q)$ taking finitely many values in $\mathbb{Q} \cap (0, 1/2]$, larger than the Potts critical temperature, for which the measure arising by decimation of the q-Potts model is not consistent with any quasilocal specification; in particular, it is not Gibbsian.

4.3. Non-Gibbsianness for an Interval of Temperatures Above T_c ($d \geqslant 3$)

The limitations of the method of the previous section (finite sequence of particular temperatures) can be overcome by choosing the decimated spins in a *random* fashion, for instance, 2 with probability f and 1 otherwise. By using a random version of Pirogov–Sinai due to Zahradník[32] we can then prove the analogue of Theorem 4.2 for a whole interval of temperatures above T_c. Zahradník's proof of the existence of coexisting phases for random systems only applies for small disorder (f small) and dimensions $d \geqslant 3$.

This part of the argument is technically complicated, but is essentially identical to the one given in ref. 30, pp. 1012–1013, for the Ising model, except that for Potts models $1/q$ plays the role of the temperature in low-temperature Ising models and the temperature plays the role of the magnetic field. We skip the details and content ourselves with stating the conclusions.

Theorem 4.3. For each dimension $d \geqslant 3$ and each decimation period N there exists a q_0 such that for each $q > q_0$ there exists a non-empty interval of temperatures $(T_c, T(q))$ where the measure arising from the decimation of the q-Potts model is not consistent with any quasilocal specification; in particular, it is not Gibbsian. The temperatures $T(q)$ increase with q.

5. CONCLUSIONS AND FINAL COMMENTS

We have shown examples of renormalization transformations exhibiting pathologies deep inside the one-phase region and (for the first time) within the high-temperature phase. These examples suggest that the occurrence of these types of pathologies is a rather robust phenomenon. It is still not clear, however, what the practical consequences of these pathologies are.

A natural question is the size of the set of "pathological" configurations ω'_{special} at which some finite-volume conditional probability is non-quasilocal (discontinuous). In the case of the majority rule acting on the Ising model in a strong field, this set of pathological configurations is of measure zero with respect to the (unique) Ising Gibbs state. This follows from the results of ref. 8. The same is true for the case of block averaging in a field (analyzed in ref. 30, p. 1014). This raises the possibility of restoring a weak form of Gibbsianness defined only almost surely.[1, 23, 25, 7, 18]

For the high-temperature pathologies of the decimated Potts models, we expect them to disappear if the decimation transformation is repeated sufficiently many times. Alternatively, for any temperature above T_c the pathologies should be absent if the decimation is taken with linear period N large enough. This expectation is based on similar results obtained by Martinelli and Olivieri[26] for the Ising model in nonzero field (which is the analogue of $T > T_c$ for the Potts-model transition). On the other hand, for any fixed N our Theorem 4.3 implies that for q large enough every open interval around the transition temperature T_c includes (a whole subinterval of) temperatures where the decimation transformation produces non-Gibbsianness. This is to be contrasted with some results[21, 1, 31] suggesting an opposite conclusion for neighborhoods of the critical temperature of the Ising model. Although the arguments presented in these works are not completely rigorous—they are based on numerical studies of a small number of decimated configurations—one may indeed expect differences between the cases for which there is a continuous phase transition at T_c (low-q Potts models) and the cases where the phase transition at T_c is of first order (the high-q Potts models analyzed here).

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